PERTURBATION ANALYSIS OF A STRONGLY VAPORIZING ROTATING PARTICLE IN A LINEAR TEMPERATURE FIELD

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ABSTRACT: For many situations of interest, the gasification of solid particles or liquid drops by evaporation or other chemical reaction leads to strong radial fields. As a problem of fundamental interest, we are analyzing the case when such particles rotate slowly while producing gas at a significantly higher rate. In this range of approximation, considerable analytical progress is possible through a perturbation expansion with the rotational Reynolds number being a small parameter. The radial Reynolds number is considered to be $O(1)$. In previous work on this class of problems, solution sets have been developed and the analysis has led to a class of polynomials that are particularly useful for perturbation analysis of such problems. In the current development, a solution is obtained for the case when the vaporizing particle is rotating in a gaseous medium while in a linear external temperature field where the rotational axis is perpendicular to the linear field. The problem is three-dimensional, and the regular perturbation analysis in the spherical coordinate system has led to analytical expressions for the temperature field.

Nomenclature

- $k$: thermal conductivity
- $Nu$: Nusselt number
- $p$: pressure
- $p_0$: primary flow pressure
- $Pe = Re_R Pr$: Peclet number
- $Pr$: Prandtl number
- $q$: heat flux
- $q_0$: far-field heat flux
- $R$: particle radius
- $r$: radial coordinate
- $\hat{r}$: radial unit vector
- $Re_R$: radial Reynolds number
- $R_0(x)$: polynomials defined by [5]
- $T(r, \theta, \phi)$: temperature field
- $U_0 = \Omega R$: secondary velocity scale
- $U_R$: primary velocity scale
- $u$: velocity
- $u_0$: primary radial flow
- $x, y$: coordinates
- $r$ (prime): secondary flow quantities
- derivatives
1 INTRODUCTION

For fluid or solid particles undergoing phase change at the surface, the radial velocity produced can be quite large due to the significantly lower density of the gas produced. This can also be the case for gas-producing chemical reaction at the particle surface. There have been numerous problems of fundamental interest in heat and mass transfer that have been solved by perturbation methods under various asymptotic limits. For example, the fluid-mechanical problem of condensation on a moving liquid drop (Sadhal & Ayyaswamy [6]) has been handled by treating the radially symmetric problem as the leading order and introducing the translation as a perturbation. This procedure has led to a regular-perturbation solution that has provided considerable insight into the fundamental fluid mechanics concerning the drag on the drop. The heat and mass transfer analysis for such problems was first carried out by Chung, Ayyaswamy & Sadhal [1] in relation to rapid condensation on moving drops. For the situation of droplet evaporation and combustion, Gogos et al [2] have developed singular perturbation solutions for a complex set of equations dealing with fluid mechanics coupled with vapor transport, chemical reaction (combustion), and heat transfer. Further work in this range problems has been conducted by Jog, Ayyaswamy & Cohen [3] who took the droplet combustion problem to higher order. In the situation of a vaporizing drop in an electric field, Nguyen & Chung [4] have utilized the development of Sadhal & Ayyaswamy [6] to obtain the temperature field. The fundamentals of the fluid mechanics for this class of problems have been treated in detail from a rigorous mathematical standpoint by Sadhal [5] who identified a new class of polynomials for a wide range of problems involving drops and particles with strong radial flow. All of these examples provide considerable physical insight into the fluid dynamics together with heat and mass transfer in such systems.

There has been little work in the way of non-axisymmetric situations in this class of problems, especially concerning heat and mass transfer. Sadhal [5] developed some pure fluid flow solutions for a particle in linear shear, and also provided some corresponding fundamental singularities in fluid mechanics. The asymmetric problems present analytical challenges, and to tackle this class of problems, we start with a rotating particle in a linear far-field temperature field. Of course, the distinct feature is the strong radial field for this system.

2 ANALYSIS

The development of the flow fields for various situation involving particles with strong radial flow is given in detail by Sadhal [5]. We repeat some of the equations for the sake of clarity and completeness. For incompressible flow, we have in the steady state

\[
\frac{1}{\rho} u \cdot \nabla u + \nabla p = \nu \nabla^2 u, \quad \nabla \cdot u = 0. \tag{1}
\]

The temperature field is described by

\[
\frac{1}{\alpha} u \cdot \nabla T = \nabla^2 T. \tag{2}
\]
In the spherical coordinate system with the origin at the center of the sphere, the boundary conditions for fluid flow are as follows:

\[
\begin{align*}
\mathbf{u}|_{r=R} &= U_R \hat{r} + \Omega R \sin \theta \hat{\phi}, \\
\mathbf{u}|_{r \to \infty} &= 0
\end{align*}
\]

where \( U_R \) is the radial velocity at the particle surface due to evaporation, sublimation, or other chemical reaction, and \( \Omega \) is the angular velocity of the particle. For the temperature field, we have

\[
\begin{align*}
T|_{r=R} &= T_s, \\
T|_{r \to \infty} &= T_\infty + \frac{q_0}{k} y = T_\infty + \frac{q_0}{k} \sin \theta \sin \phi,
\end{align*}
\]

where \( q_0 \) represents the far-field thermal flux for the undisturbed system when there is no particle.

At the leading order, we have a radially dominant spherically symmetric flow,

\[
\mathbf{u}_0 = U_R \left( \frac{R^2}{r^2} \right) \hat{r}, \quad p_0 = -\frac{1}{2} \rho U_R^2 \left( \frac{R^4}{r^4} \right)
\]

A secondary flow, \( \mathbf{u}' \), is considered as a perturbation so that

\[
\mathbf{u} = \mathbf{u}_0 + \mathbf{u}',
\]

and

\[
p = p_0 + p'.
\]

Likewise, for the temperature field,

\[
T = T_0 + T'.
\]

We now apply \( U_R \) as a radial velocity scale of the primary flow, and \( U_0 = \Omega R \) as the characteristic velocity of the secondary flow. The following non-dimensional parameters are adopted (see Sadhal [5]):

\[
\begin{align*}
\mathbf{u}^* &= \frac{\mathbf{u} R}{v}, & \mathbf{u}_0^* &= \frac{\mathbf{u}_0}{U_R}, & \mathbf{u}^* &= \frac{\mathbf{u}}{\Omega R}, & r^* &= \frac{r}{R}, \\
\varepsilon &= \frac{\Omega R^2}{v}, & Re_R &= \frac{U_R R}{v}, & p_0^* &= \frac{p_0 R}{\mu U_R}, & p^* &= \frac{pR^2}{\mu v}, & p' &= \frac{p'}{\Omega \mu}, \\
T^* &= \frac{T - T_\infty}{T_s - T_\infty}, & Pr &= \frac{v}{\alpha}, & Pe &= Re_R Pr, & Nu_x &= \frac{q_0 R}{k (T_s - T_\infty)}, & \nabla^* &= R \nabla.
\end{align*}
\]

where \( R \) is the radius of the spherical particle and \( \varepsilon = \Omega R^2/v \ll 1 \) is the rotational Reynolds number. With the nondimensionalization, Equations (8) and (9) become

\[
\begin{align*}
\mathbf{u}^* &= Re_R \mathbf{u}_0^* + \varepsilon \mathbf{u}^*, \\
p^* &= Re_R p_0^* + \varepsilon p^*.
\end{align*}
\]

and

\[
T^* = T_0^* + \varepsilon T'^*.
\]

The continuity and the momentum equations take the following forms [5]

\[
\nabla \cdot (Re_R \mathbf{u}_0 + \varepsilon \mathbf{u}') = 0,
\]

\[
Re_R^2 \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + Re_R \varepsilon (\mathbf{u}_0 \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{u}_0) + \varepsilon^2 \mathbf{u}' \cdot \nabla \mathbf{u}' + Re_R \nabla p_0 + \varepsilon \nabla p' = Re_R \nabla^2 \mathbf{u}_0 + \varepsilon \nabla^2 \mathbf{u}'.
\]

where the asterisks have been dropped. Employing the perturbation scheme [5],

\[
\begin{align*}
\mathbf{u} &= Re_R \mathbf{u}_0 + \varepsilon \mathbf{u}' = Re_R \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots, \\
p &= Re_R p_0 + \varepsilon p' = Re_R p_0 + \varepsilon \mathbf{p}_1 + \varepsilon^2 \mathbf{p}_2 + \cdots, \\
T &= T_0 + \varepsilon T' = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots.
\end{align*}
\]
2.1 Leading-Order Solution $O(\varepsilon^0)$

For the fluid-flow problem, the leading order $[O(\varepsilon^0)]$, the solution is, of course, the purely radial flow,

$$u_0 = \frac{1}{r^2} \hat{r}$$

(20)

For the heat-transfer part, to the leading order, the energy equation (2) can be expressed as

$$\frac{\partial^2 T_0}{\partial r^2} + \left( \frac{2}{r} - \frac{Pe}{r^2} \right) \frac{\partial T_0}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T_0}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T_0}{\partial \phi^2} = 0,$$

(21)

or in terms of $\bar{\mu} = \cos \theta$,

$$\frac{\partial^2 T_0}{\partial r^2} + \left( \frac{2}{r} - \frac{Pe}{r^2} \right) \frac{\partial T_0}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \bar{\mu}} \left[ (1 - \bar{\mu}^2) \frac{\partial T_0}{\partial \bar{\mu}} \right] + \frac{1}{r^2 (1 - \bar{\mu}^2)} \frac{\partial^2 T_0}{\partial \phi^2} = 0.$$  

(22)

The solution for Equation (22) can be expressed in terms of eigenfunctions

$$T_0(r, \bar{\mu}, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_{0mn}(r) P_n^m(\bar{\mu}) e^{im\phi},$$

where $P_n^m(\bar{\mu})$ denotes associated Legendre functions, and $T_{0mn}(r)$ satisfies the ordinary differential equation,

$$\frac{d^2 T_{0mn}}{dr^2} + \left( \frac{2}{r} - \frac{Pe}{r^2} \right) \frac{d T_{0mn}}{dr} - \frac{n(n + 1)}{r^2} T_{0mn}(r) = 0.$$  

(24)

If we let $x = r/Pe$, we can express Equation (24) as

$$\frac{d^2 T_{0mn}}{dx^2} + \left( \frac{2}{x} - \frac{1}{x^2} \right) \frac{d T_{0mn}}{dx} - \frac{n(n + 1)}{x^2} T_{0mn}(x) = 0,$$

(25)

which has the general solution,

$$T_{0mn}(x) = A_{mn} e^{-1/x} R_n(x) + B_{mn} R_n(-x),$$

(26)

where $R_n(x)$ is a set of polynomials (see Sadhal [5]),

$$R_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} x^k, \quad n = 0, 1, 2, 3, \ldots$$

(27)

Upon satisfying the boundary conditions at $r = 1$ and $r \to \infty$, the solution reduces to

$$T_0(r, \theta, \phi) = \frac{e^{-Pe/r} - 1}{e^{-Pe} - 1} + \frac{1}{2} Nu_x Pe \left[ 1 - \frac{2}{Pe} \right] \left( 1 + \frac{2}{Pe} \right) e^{-Pe/r} - \left( 1 + \frac{2}{Pe} \right) \left( 1 - \frac{2}{Pe} \right) e^{-Pe} \sin \theta \sin \phi.$$  

(28)

2.2 First-Order Solution $O(\varepsilon)$

To the next order $[O(\varepsilon)]$, the continuity and momentum equations are

$$\nabla \cdot u_1 = 0,$$

(29)

$$Re \left( u_0 \cdot \nabla u_1 + u_1 \cdot \nabla u_0 \right) + \nabla p_1 = \nabla^2 u_1.$$  

(30)
We are now considering particle rotation as a perturbation to the purely radial leading order. This type of flow has been dealt with by Sadhal [5] who considered the primary flow is the purely radial field given by equation (7) while the secondary flow consists of perturbations arising from rotation. Here, the velocity field is given by \((u_r, u_\phi)\) consisting of a radial and an angular part. With an angular velocity \(\Omega\) of the rotating particle, we use \(U_0 = \Omega R\) as a characteristic velocity for the rotational flow. Keeping in mind the scaling [Equation (11)], the perturbation scheme given by equations (17)-(18) may be represented in dimensionless variables as

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 u_r \right) + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left( r^2 (1 - \mu^2) \frac{\partial T_1}{\partial \mu} \right) + \frac{1}{r^2 (1 - \mu^2)^{1/2}} \frac{\partial^2 T_1}{\partial \phi^2} = \frac{Pr}{r (1 - \mu^2)^{1/2}} \frac{\partial T_0}{\partial \phi} u_\phi, \tag{38}
\]

so that it takes the form,

\[
\frac{\partial^2 T_1}{\partial r^2} + \left( \frac{2}{r} - \frac{Pe}{r^2} \right) \frac{\partial T_1}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial T_1}{\partial \mu} \right] + \frac{1}{r^2 (1 - \mu^2)^{1/2}} \frac{\partial^2 T_1}{\partial \phi^2} = \frac{Pr}{r (1 - \mu^2)^{1/2}} \frac{\partial T_0}{\partial \phi} u_\phi. \tag{38}
\]

where the right-hand side is

\[
\frac{u_\phi Pr}{r (1 - \mu^2)^{1/2}} \frac{\partial T_0}{\partial \phi} = \frac{1}{2} \text{Nu}_p \frac{Pr}{Pe} \frac{r^2 e^{-Re_K/r} - (r^2 - rRe_K + \frac{1}{2} Re_K^2)}{r^2 \left[ e^{-Re_K} - (1 - Re_K + \frac{1}{2} Re_K^2) \right]} \times \frac{(1 - \frac{Pe}{Pe}) (1 + \frac{2Pe}{Pe}) e^{-Pe/r} - (1 + \frac{2Pe}{Pe}) (1 - \frac{2Pe}{Pe}) e^{-Pe}}{(1 - \frac{Pe}{Pe}) + (1 + \frac{2Pe}{Pe}) e^{-Pe}} \sin \theta \cos \phi. \tag{39}
\]

A 'general' solution can be expressed as

\[
T_1(r, \theta, \phi) = \text{Real} \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{m} T_{1mn}(r) P_n^m(\cos \theta)e^{im\phi} \right]. \tag{40}
\]
Based on the forcing function (39) which contains \( \sin \theta \cos \phi \), and noting that \( P_1^1(\cos \theta) = -\sin \theta \), we can limit the summation to \( m = 1 \) and \( n = 1 \), i.e.,

\[
T_1(r, \theta, \phi) = T_{111}(r)(-\sin \theta)\cos \phi,
\]

where \( T_{111}(r) \) satisfies

\[
\frac{d^2 T_{111}(r)}{dr^2} + \left( \frac{2}{r} - \frac{Pe}{r^2} \right) \frac{dT_{111}(r)}{dr} - \frac{2}{r^2} T_{111}(r) = F_1(r),
\]

with boundary conditions,

\[
T_{111}(1) = 0 \quad \text{and} \quad T_{111}(r)\big|_{r\to\infty} = 0.
\]

The solution is found by variation of parameters as

\[
T_{111}(r) = \left[ C_{11} + G_1(r) \right] \left[ e^{-Pe/r} \left( 1 + \frac{2r}{Pe} \right) + \left( 1 - \frac{2r}{Pe} \right) \right] + G_2(r) \left( 1 - \frac{2r}{Pe} \right),
\]

where

\[
G_1(r) = \int_1^r \frac{\xi^2}{Pe} \left( 1 - \frac{2\xi}{Pe} \right) e^{Pe/\xi} F_1(\xi) \, d\xi
\]

and

\[
G_2(r) = \int_r^\infty \frac{\xi^2}{Pe} \left( 1 + \frac{2\xi}{Pe} \right) + \left( 1 - \frac{2\xi}{Pe} \right) e^{Pe/\xi} F_1(\xi) \, d\xi
\]

with \( F_1(\xi) \) defined by Equation (39). The constants \( C_{11} \) and \( D_{11} \) are determined by the boundary conditions given by Equation (43), leading to the following

\[
C_{11} = -\frac{G_2(1) \left( 1 - \frac{2}{Pe} \right)}{e^{-Pe} \left( 1 + \frac{2}{Pe} \right) + \left( 1 - \frac{2}{Pe} \right)}.
\]

The expression for the \( O(\varepsilon) \) temperature field is

\[
T_1(r, \theta, \phi) = \left\{ [C_{11} + G_1(r)] \left[ e^{-Pe/r} \left( 1 + \frac{2r}{Pe} \right) + \left( 1 - \frac{2r}{Pe} \right) \right] + G_2(r) \left( 1 - \frac{2r}{Pe} \right) \right\} (-\sin \theta)\cos \phi,
\]

or in dimensional quantities,

\[
T_1(r, \theta, \phi) = T_\infty - (T_s - T_\infty) \left\{ [C_{11} + G_1(r)] \left[ e^{-PeR/r} \left( 1 + \frac{2r}{PeR} \right) + \left( 1 - \frac{2r}{PeR} \right) \right] + G_2(r) \left( 1 - \frac{2r}{PeR} \right) \right\} \sin \theta \cos \phi.
\]

The complete temperature field to \( O(\varepsilon) \) is given by the first two terms in Equation (19).

### 3 RESULTS AND DISCUSSION

The temperature field has been numerically evaluated, and separate isotherm maps for \( T_0(r, \theta, \phi) \), \( T_1(r, \theta, \phi) \), and \( T(r, \theta, \phi) = T_0 + \varepsilon T_1 \) in the equatorial plane are given in Figure 1 for \( Pe = 1 \) and \( Pr = 0.7 \). For higher values of the Péclet number, \( Pe = 3, 5 \), similar maps are given in Figures 2 and 3. Here the symmetry is broken by the combined radial flow and the linear temperature field far away. The particle rotation introduces additional asymmetry, as seen in left versus right halves in Figures 1(c)-3(c). These effects increase with increasing \( Pe \). Besides the temperature field, the heat-transfer rate is of physical interest. The conductive heat flux at the
Figure 1: Isotherm map on the equatorial plane for $Pr = 0.7$ and $Pe = 1$: (a) $T_0$; (b) $T_1$ and (c) $T_0 + \epsilon T_1$

Figure 2: Isotherm map on the equatorial plane for $Pr = 0.7$ and $Pe = 3$: (a) $T_0$; (b) $T_1$ and (c) $T_0 + \epsilon T_1$

Figure 3: Isotherm map on the equatorial plane for $Pr = 0.7$ and $Pe = 5$: (a) $T_0$; (b) $T_1$ and (c) $T_0 + \epsilon T_1$

surface of the sphere, in dimensional form, is given by

$$
q_{\text{cond}}' = -k \frac{\partial T}{\partial r} \bigg|_{r=R} \\
= -k \frac{\partial}{\partial r} (T_0 + \epsilon T_1) \bigg|_{r=R} \\
= k (T_s - T_\infty) \frac{Pe}{R} \left( \frac{e^{-Pe}}{1 - e^{-Pe}} \right) - \frac{1}{2} \frac{q_0 Pe^2}{1 - e^{-Pe}} \left[ e^{-Pe} \left( 1 + \frac{2 Pe}{R e} \right) \right] \sin \theta \sin \phi \\
+ \frac{k}{R} (T_s - T_\infty) \left\{ C_{11} + G_1(1) \right\} e^{-Pe} \left( Pe + 2 \frac{2 Pe}{R e} \right) - \left\{ C_{11} + G_2(1) \right\} e^{-Pe} \left( \frac{2 Pe}{R e} \right) \sin \theta \cos \phi.
$$

Integrating this over the whole surface of the sphere, we obtain

$$
Q_{\text{cond}} = 4\pi R k (T_s - T_\infty) \left( \frac{Pe e^{-Pe}}{1 - e^{-Pe}} \right),
$$

which is the contribution from only the spherically symmetric part. It is interesting to note that $Q_{\text{cond}}$ diminishes with increasing $Pe$. However, we need to remember that this is only the contribution from conduction. In this
case, there is gas being emitted from the surface of the sphere that needs to be accounted for. Putting in this additional contribution yields [5]

\[
Q_{\text{tot}} = 4\pi R k (T_s - T_\infty) \left[ \left( \frac{Pe e^{-Pe}}{1 - e^{-Pe}} \right) + \frac{Pe}{1 - e^{-Pe}} R U_R \right]
\]

\[
= 4\pi R k (T_s - T_\infty) \left( \frac{Pe}{1 - e^{-Pe}} \right)
\]

which has, for large Pe a linear behavior with Pe. The same result can be derived if \(Q_{\text{cond}}\) is calculated as \(r \to \infty\). In that region, the temperature is approaching \(T_\infty\) and convective contribution vanishes while still maintaining overall heat balance in the steady state. A plot of \(Q_{\text{tot}}\) versus \(Pe\) is provided in Figure 4(a).

\[
\begin{align*}
Q_{\text{y,cond}} &= -\frac{1}{2} q_0 R^2 Pe^2 \int_0^\pi \int_0^\pi \left[ \frac{-Pe}{1 - Pe} + Pe \left( 1 + \frac{2}{Pe} \right) \right] \sin^2 \theta \sin \phi \, d\theta \, d\phi \\
&= -\frac{1}{2} q_0 \pi R^2 Pe^2 \left[ \frac{-Pe}{1 - Pe} + Pe \left( 1 + \frac{2}{Pe} \right) \right]
\end{align*}
\]

(53)

In the \(x\)-direction, we have the \(\cos \phi\) term leading to a nonzero value,

\[
\begin{align*}
Q_{\text{x,cond}} &= \varepsilon k R (T_s - T_\infty) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi \left[ \frac{1}{2} Pe \left( Pe + 2 + \frac{2}{Pe} \right) \right] \sin^2 \theta \cos \phi \, d\theta \, d\phi \\
&= \varepsilon k R \pi (T_s - T_\infty) \left[ C_{11} + G_1(1) \right] e^{-Pe} \left( Pe + 2 + \frac{2}{Pe} \right) \\
&- \left[ C_{11} + G_2(1) \right] e^{-Pe} \left( \frac{2}{Pe} \right) \\
&= \varepsilon k R \pi (T_s - T_\infty) \left[ C_{11} + G_1(1) \right] e^{-Pe} \left( Pe + 2 + \frac{2}{Pe} \right) \\
&- \left[ C_{11} + G_2(1) \right] e^{-Pe} \left( \frac{2}{Pe} \right)
\end{align*}
\]

(54)
4 CONCLUDING REMARKS

Our treatment of the heat-transfer problem of a rotating particle with a strong radial flow has provided fully analytical results for the temperature field. The perturbation method is an effective tool for analytical calculation when dealing with heat transfer and fluid flow problems. The technique used in this work provides results that give considerable insight into the physics of the system. Strong radial flow due to evaporation or combustion is a common occurrence and mathematical methods to get a deep understanding of the physical phenomena are particularly useful. While we have obtained various heat-flow results that provide meaningful description the overall thermal transport, it should be noted that the perturbed temperature field \( T_1 (r, \theta, \phi) \) given in Equation (49) does not vanish as \( r \to \infty \) but goes to a constant value. However, with the leading order being linear in \( y \) the relative error due to this singular behavior is relatively small, and not problematic for heat-flow calculations which involve derivatives of the temperature field. Nevertheless, this does indeed leave the door open for more sophisticated development using singular perturbation analysis.

References


